

ON A LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE HIGHER ORDER IN TWO VARIABLES WITH INITIAL CONDITION WHOSE COEFFICIENTS ARE REAL-VALUED SIMPLE STEP FUNCTIONS

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Abstract

By using the method developed in the paper [*Georg. Inter. J. Sci. Tech.*, Volume 3, Issue 1 (2011), 107-129], it is obtained a representation in an explicit form of the weak solution of a linear partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions

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1. Introduction

In [5] has been obtained a representation in an explicit form of the solution of the linear partial differential equation of the higher order in two variables with initial condition whose coefficients were real-valued coefficients. The aim of the present manuscript is resolve an analogous problem for a linear partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions.

The paper is organized as follows.

In Section 2, we consider some auxiliary results obtained in the paper [5]. In Section 3, it is obtained a representation in an explicit form of the weak solution of the partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions.

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2. Some auxiliary results

Definition 2.1. Fourier differential operator $(\mathcal{F}) \frac{\partial}{\partial x}$ in R^∞ is defined as follows :

$$(\mathcal{F}) \frac{\partial}{\partial x} \begin{pmatrix} \frac{a_0}{2} \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1\pi}{l} & 0 & 0 & 0 & 0 & \dots \\ 0 & -\frac{1\pi}{l} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{2\pi}{l} & 0 & 0 & \dots \\ 0 & 0 & 0 & -\frac{2\pi}{l} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi}{l} & \ddots \\ 0 & 0 & 0 & 0 & 0 & -\frac{3\pi}{l} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} \frac{a_0}{2} \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \end{pmatrix}. \quad (2.1)$$

For $n \in \mathbb{N}$, let $FD^n[-l, l[$ be a vector space of all n -times differentiable functions on $[-l, l[$ such that for arbitrary $0 \leq k \leq n-1$, a series obtained by a differentiation term by term of the Fourier series of $f^{(k)}$ pointwise converges to $f^{(k+1)}$ for all $x \in [-l, l[$.

Lemma 2.2. Let $f \in FD^{(1)}[-l, l[$. Let G_M be an embedding of the $FD^{(1)}[-l, l[$ in to R^∞ which sends a function to a sequence of real numbers consisting from its Fourier coefficients. i.e., if

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \quad (x \in [-l, l[),$$

then $G_F(f) = (\frac{c_0}{2}, c_1, d_1, c_2, d_2, \dots)$. Then, for $f \in FD^{(1)}[-l, l[$, the following equality

$$\left(G_F^{-1} \circ (\mathcal{F}) \frac{\partial}{\partial x} \circ G_F\right)(f) = \frac{\partial}{\partial x}(f) \quad (2.2)$$

holds.

Proof. Assume that for $f \in FD^{(1)}[-l, l[$, we have the following representation

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \quad (x \in [-l, l[).$$

By the definition of the class $FD^{(1)}[-l, l[$, we have

$$\begin{aligned} \frac{d}{dx}(f) &= \frac{\partial}{\partial x} \left(\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \right) = \\ &= \sum_{k=1}^{\infty} c_k \frac{\partial}{\partial x} \left(\cos\left(\frac{k\pi x}{l}\right) \right) + d_k \frac{\partial}{\partial x} \left(\sin\left(\frac{k\pi x}{l}\right) \right) = \\ &= \sum_{k=1}^{\infty} -c_k \frac{k\pi}{l} \sin\left(\frac{k\pi x}{l}\right) + d_k \frac{k\pi}{l} \cos\left(\frac{k\pi x}{l}\right) = \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{k\pi d_k}{l} \cos\left(\frac{k\pi x}{l}\right) - \frac{k\pi c_k}{l} \sin\left(\frac{k\pi x}{l}\right).$$

By the definition of the composition of mappings, we have

$$\begin{aligned} \left(G_F^{-1} \circ (\mathcal{F}) \frac{\partial}{\partial x} \circ G_F\right)(f) &= G_F^{-1}((\mathcal{F}) \frac{\partial}{\partial x}((G_F(f)))) = G_F^{-1}((\mathcal{F}) \frac{\partial}{\partial x} \begin{pmatrix} \frac{c_0}{2} \\ c_1 \\ d_1 \\ c_2 \\ d_2 \\ c_3 \\ d_3 \\ \vdots \end{pmatrix}) = \\ &= G_F^{-1} \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1\pi}{l} & 0 & 0 & 0 & 0 & \dots \\ 0 & -\frac{1\pi}{l} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{2\pi}{l} & 0 & 0 & \dots \\ 0 & 0 & 0 & -\frac{2\pi}{l} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi}{l} & \ddots \\ 0 & 0 & 0 & 0 & 0 & -\frac{3\pi}{l} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} \frac{c_0}{2} \\ c_1 \\ d_1 \\ c_2 \\ d_2 \\ c_3 \\ d_3 \\ \vdots \end{pmatrix} \right) = \\ &= G_F^{-1} \left(\begin{pmatrix} 0 \\ \frac{1\pi d_1}{l} \\ -\frac{1\pi c_1}{l} \\ \frac{2\pi d_2}{l} \\ -\frac{2\pi c_2}{l} \\ \frac{3\pi d_3}{l} \\ -\frac{3\pi c_3}{l} \\ \vdots \end{pmatrix} \right) = \sum_{k=1}^{\infty} \frac{k\pi d_k}{l} \cos\left(\frac{k\pi x}{l}\right) - \frac{k\pi c_k}{l} \sin\left(\frac{k\pi x}{l}\right). \end{aligned}$$

□

By the scheme used in the proof of Lemma 2.1, we can get the validity of the following assertion.

Lemma 2.3. *Let G_M be an embedding of the $FD^n[-l, l[$ in to R^∞ which sends a function to a sequence of real numbers consisting from its Fourier coefficients.*

Then, for $f \in FD^{(n)}[-l, l[$ and $A_k \in R(0 \leq k \leq n)$, the following equality

$$\left(G_F^{-1} \circ \left(\sum_{k=0}^n A_k ((\mathcal{F}) \frac{\partial}{\partial x})^k\right) \circ G_F\right)(f) = \sum_{k=0}^n a_k \frac{\partial^k}{\partial x^k}(f) \quad (2.3)$$

holds.

Example 2.4. [3] If A is the real matrix

$$\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}, \quad (2.4)$$

then

$$e^{tA} = e^{\sigma t} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}. \quad (2.5)$$

Lemma 2.5. For $m \geq 1$, let us consider a linear autonomous nonhomogeneous ordinary differential equations of the first order

$$\frac{d}{dt}((a_k)_{k \in \mathbb{N}}) = \left(\sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n \right) \times ((a_k)_{k \in \mathbb{N}}) + (f_k)_{k \in \mathbb{N}} \quad (2.6)$$

with initial condition

$$(a_k(0))_{k \in \mathbb{N}} = (C_k)_{k \in \mathbb{N}}, \quad (2.7),$$

where

(i) $(C_k)_{k \in \mathbb{N}} \in \mathbf{R}^\infty$;

(ii) $f = (f_k)_{k \in \mathbb{N}}$ is the sequence of continuous functions of a parameter t on R .

For each $k \geq 1$, we put

$$\sigma_k = \sum_{n=0}^m (-1)^n A_{2n} \left(\frac{k\pi}{l} \right)^{2n}, \quad (2.8)$$

$$\omega_k = \sum_{n=0}^{m-1} (-1)^n A_{2n+1} \left(\frac{k\pi}{l} \right)^{2n+1}. \quad (2.9)$$

Then the solution of (2.6)-(2.7) is given by

$$(a_k(t))_{k \in \mathbb{N}} = e^{t(\sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n)} \times (C_k)_{k \in \mathbb{N}} + \int_0^t e^{(\tau-t)(\sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n)} \times f(\tau) d\tau, \quad (2.10)$$

where $e^{t(\sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n)}$ denotes an exponent of the matrix $t(\sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n)$ and it exactly coincides with an infinite-dimensional $(1, 2, 2, \dots)$ -cellular matrix $D(t)$ with cells $(D_k(t))_{k \in \mathbb{N}}$ for which $D_0(t) = (e^{tA_0})$ and

$$D_k(t) = e^{\sigma_k t} \begin{pmatrix} \cos(\omega_k t) & \sin(\omega_k t) \\ -\sin(\omega_k t) & \cos(\omega_k t) \end{pmatrix}, \quad (2.11)$$

where for $k \geq 1$, σ_k and ω_k are defined by (2.8)-(2.9), respectively.

Proof. We know that if we have a linear autonomous inhomogeneous ordinary differential equations of the first order

$$\frac{d}{dt}((a_k)_{k \in \mathbb{N}}) = E \times ((a_k)_{k \in \mathbb{N}}) + (f_k)_{k \in \mathbb{N}} \quad (2.12)$$

with initial condition

$$(a_k(0))_{k \in \mathbb{N}} = (C_k)_{k \in \mathbb{N}}, \quad (2.13)$$

where

- (i) $(C_k)_{k \in \mathbb{N}} \in \mathbf{R}^\infty$;
 - (ii) $(f_k)_{k \in \mathbb{N}}$ is the sequence of continuous functions of parameter t on R ;
 - (iii) E is an infinite dimensional $(1, 2, 2, \dots)$ -cellular matrix with cells $(E_k)_{k \in \mathbb{N}}$.
- Then the solution of (2.6)-(2.7) is given by (cf. [3], §6, Section 1)

$$(a_k(t))_{k \in \mathbb{N}} = e^{tE} \times (C_k)_{k \in \mathbb{N}} + \int_0^t e^{(\tau-t)E} \times f(\tau) d\tau, \quad (2.14)$$

where e^{tE} and $e^{(\tau-t)E}$ denote exponents of matrices tE and $(\tau-t)E$, respectively.

Note that $t \sum_{n=0}^{2m} A_n \left((\mathcal{F}) \frac{\partial}{\partial x} \right)^n$ is an infinite-dimensional $(1, 2, 2, \dots)$ -cellular matrix with cells $(tE_k)_{k \in \mathbb{N}}$ such that $tE_0 = (tA_0)$ and

$$tE_k = \begin{pmatrix} t\sigma_k & t\omega_k \\ -t\omega_k & t\sigma_k \end{pmatrix} \quad (2.15)$$

for $k \geq 1$. Under notations (2.8)-(2.9), by using Example 2.4 we get that for $t \in R$, e^{tE} exactly coincides with an infinite-dimensional $(1, 2, 2, \dots)$ -cellular matrix $D(t)$ with cells $(D_k(t))_{k \in \mathbb{N}}$ for which $D_0(t) = (e^{tA_0})$ and

$$D_k(t) = e^{\sigma_k t} \begin{pmatrix} \cos(\omega_k t) & \sin(\omega_k t) \\ -\sin(\omega_k t) & \cos(\omega_k t) \end{pmatrix}. \quad (2.16).$$

Note that, for $0 \leq \tau \leq t$, the matrix $e^{(\tau-t)E}$ exactly coincides with an infinite-dimensional $(1, 2, 2, \dots)$ -cellular matrix $D(\tau-t)$. □

The following proposition is a simple consequence of Lemma 2.5.

Corollary 2.6. *For $m \geq 1$, let us consider a linear partial differential equation*

$$\frac{\partial}{\partial t} \Psi(t, x) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \Psi(t, x) \quad ((t, x) \in [0, +\infty[\times [-l, l]) \quad (2.17)$$

with initial condition

$$\Psi(0, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \in FD^{(0)}[-l, l]. \quad (2.18)$$

If $(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \dots)$ is such a sequence of real numbers that a series $\Psi(t, x)$ defined by

$$\Psi(t, x) = \frac{e^{tA_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t} \left((c_k \cos(\omega_k t) + d_k \sin(\omega_k t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k \cos(\omega_k t) - c_k \sin(\omega_k t)) \sin\left(\frac{k\pi x}{l}\right) \right) \quad (2.19)$$

belongs to the class $FD^{(2m)}[-l, l]$ as a series of a variable x for all $t \geq 0$, and is differentiable term by term as a series of a variable t for all $x \in [-l, l]$, then Ψ is a solution of (2.17)-(2.18).

3. Solution of a linear partial differential equation of the higher order in two variables with initial condition when coefficients are real-valued simple step functions

Let $0 = t_0 < \dots < t_I = T$ and $-l = x_0 < \dots < x_J = l$. Suppose that

$$A_n(t, x) = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} A_n^{(i,j)} \times \chi_{[t_i, t_{i+1}[\times [x_j, x_{j+1}[}(t, x).$$

For $m \geq 1$, let us consider a partial differential equation

$$\frac{\partial}{\partial t} \Psi(t, x) = \sum_{n=0}^{2m} A_n(t, x) \frac{\partial^n}{\partial x^n} \Psi(t, x) \quad ((t, x) \in [0, T[\times [-l, l]) \quad (3.1)$$

with initial condition

$$\Psi(0, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \in FD^{(0)}[-l, l]. \quad (3.2)$$

Definition 3.1. We say that $\Psi(t, x)$ is a weak solution of (3.1)-(3.2) if the following conditions hold:

- (i) $\Psi(t, x)$ satisfies (3.1) for each $(t, x) \in [0, T[\times [-l, l[$ for which $t \neq t_i (0 \leq i \leq I)$ or $x \neq x_j (0 \leq j \leq J)$;
- (ii) $\Psi(t, x)$ satisfies (3.2);
- (iii) for each fixed $x \in [-l, l]$, the function $\Psi(t, x)$ is continuous with respect to $t \in [0, T[$, and for each $t \in [0, T[$ the function $\Psi(t, x)$ is continuous with respect to x on $[-l, l]$ except points $\{x_j : 0 \leq j \leq J-1\}$.

First, let fix j and consider a partial differential equation

$$\frac{\partial}{\partial t} \Psi_{(0,j)}(t, x) = \sum_{n=0}^{2m} A_n^{(0,j)}(t, x) \frac{\partial^n}{\partial x^n} \Psi_{(0,j)}(t, x) \quad ((t, x) \in [0, +\infty[\times [-l, l]) \quad (0,j)(PDE)$$

with initial condition

$$\begin{aligned} \Psi_{(0,j)}(t_0, x) &= \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) = \\ \frac{c_0^{(0,j)}}{2} + \sum_{k=1}^{\infty} c_k^{(0,j)} \cos\left(\frac{k\pi x}{l}\right) + d_k^{(0,j)} \sin\left(\frac{k\pi x}{l}\right) &\in FD^{(0)}[-l, l], \end{aligned} \quad (0,j)(IC)$$

By Corollary 2.6, under some restrictions on $(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \dots)$, a series $\Psi_{(0,j)}(t, x)$ defined by

$$\begin{aligned} \Psi_{(0,j)}(t, x) &= \frac{e^{tA_0^{(0,j)}} c_0^{(0,j)}}{2} + \sum_{k=1}^{\infty} e^{\sigma_k^{(0,j)} t} \left((c_k^{(0,j)} \cos(\omega_k^{(0,j)} t) + \right. \\ &\left. d_k^{(0,j)} \sin(\omega_k^{(0,j)} t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k^{(0,j)} \cos(\omega_k^{(0,j)} t) - c_k^{(0,j)} \sin(\omega_k^{(0,j)} t)) \sin\left(\frac{k\pi x}{l}\right) \right) \end{aligned} \quad (3.3)$$

is a solution of $(0, j)(PDE)$ - $(0, j)(IC)$,

Now let consider a partial differential equation

$$\frac{\partial}{\partial t} \Psi_{(1,j)}(t, x) = \sum_{n=0}^{2m} A_n^{(1,j)}(t, x) \frac{\partial^n}{\partial x^n} \Psi_{(1,j)}(t, x) \quad ((t, x) \in [0, +\infty[\times [-l, l]) \quad (1, j)(PDE)$$

with initial condition

$$\Psi_{(1,j)}(t_1, x) = \Psi_{(0,j)}(t_1, x) \quad (1, j)(IC)$$

We will try to present the solution of the $(1, j)(PDE)$ by the following form

$$\Psi_{(1,j)}(t, x) = \frac{e^{tA_0^{(1,j)}} c_0^{(1,j)}}{2} + \sum_{k=1}^{\infty} e^{\sigma_k^{(1,j)} t} \left((c_k^{(1,j)} \cos(\omega_k^{(1,j)} t) + d_k^{(1,j)} \sin(\omega_k^{(1,j)} t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k^{(1,j)} \cos(\omega_k^{(1,j)} t) - c_k^{(1,j)} \sin(\omega_k^{(1,j)} t)) \sin\left(\frac{k\pi x}{l}\right) \right) \quad (3.4)$$

In order to get validity of the condition $(1, j)(IC)$, we consider the following infinite system of equations:

$$\frac{e^{t_1 A_0^{(1,j)}} c_0^{(1,j)}}{2} = \frac{e^{t_1 A_0^{(0,j)}} c_0^{(0,j)}}{2}, \quad (3.5)$$

$$e^{\sigma_k^{(1,j)} t_1} (c_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) + d_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1)) =$$

$$e^{\sigma_k^{(0,j)} t_1} (c_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) + d_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)) \quad (k \in \mathbb{N}), \quad (3.6)$$

$$e^{\sigma_k^{(1,j)} t_1} (d_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) - c_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1)) = e^{\sigma_k^{(0,j)} t_1} (d_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) - c_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)) \quad (k \in \mathbb{N}). \quad (3.7)$$

We have

$$c_0^{(1,j)} = e^{t_1(A_0^{(0,j)} - A_0^{(1,j)})} c_0^{(0,j)}. \quad (3.8)$$

For $k \in \mathbb{N}$ we can rewrite equations (3.6)-(3.7) as follows:

$$c_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) + d_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1) = e^{(\sigma_k^{(0,j)} - \sigma_k^{(1,j)}) t_1} (c_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) + d_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)), \quad (3.9)$$

$$-c_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1) + d_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) = e^{(\sigma_k^{(0,j)} - \sigma_k^{(1,j)}) t_1} (d_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) - c_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)). \quad (3.10)$$

Setting

$$\mathbb{A} = e^{(\sigma_k^{(0,j)} - \sigma_k^{(1,j)}) t_1} (c_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) + d_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)) \quad (3.11)$$

and

$$\mathbb{B} = e^{(\sigma_k^{(0,j)} - \sigma_k^{(1,j)}) t_1} (d_k^{(0,j)} \cos(\omega_k^{(0,j)} t_1) - c_k^{(0,j)} \sin(\omega_k^{(0,j)} t_1)), \quad (3.12)$$

for $k \in \mathbb{N}$ we obtain

$$c_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) + d_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1) = \mathbb{A} \quad (3.13)$$

and

$$-c_k^{(1,j)} \sin(\omega_k^{(1,j)} t_1) + d_k^{(1,j)} \cos(\omega_k^{(1,j)} t_1) = \mathbb{B}. \quad (3.14)$$

It is obvious that the system of equations (3.13)-(3.14) has the unique solution which can be done as follows:

$$c_k^{(1,j)} = \mathbb{A} \cos(\omega_k^{(1,j)} t_1) - \mathbb{B} \sin(\omega_k^{(1,j)} t_1) \quad (3.15)$$

and

$$d_k^{(1,j)} = \mathbb{B} \cos(\omega_k^{(1,j)} t_1) + \mathbb{A} \sin(\omega_k^{(1,j)} t_1) \quad (3.16)$$

for $k \in \mathbb{N}$.

By Corollary 2.6, under some restrictions on $(\frac{c_0^{(1,j)}}{2}, c_1^{(1,j)}, d_1^{(1,j)}, c_2^{(1,j)}, d_2^{(1,j)}, \dots)$, the series $\Psi_{(1,j)}(t, x)$ defined by (3.4) is the solution of $(1, j)(PDE)$ -($1, j$)(IC),

It is obvious that under nice restrictions on coefficients participated in (3.1) and (3.2), we can continue our procedure step by step. Correspondingly we can construct a sequence $(\Psi_{(s,j)})_{0 \leq s \leq I-1, 1 \leq j \leq J-1}$ such that $\Psi_{(s,j)}$ satisfies a linear partial differential equation

$$\frac{\partial}{\partial t} \Psi_{(s,j)}(t, x) = \sum_{n=0}^{2m} A_n^{(k,j)}(t, x) \frac{\partial^n}{\partial x^n} \Psi_{(s,j)}(t, x) \quad ((t, x) \in [0, +\infty[\times [-l, l]) \quad (s, j)(PDE)$$

with initial condition

$$\Psi_{(s,j)}(t_s, x) = \Psi_{(s-1,j)}(t_s, x) = \frac{c_0^{(s,j)}}{2} + \sum_{k=1}^{\infty} c_k^{(s,j)} \cos\left(\frac{k\pi x}{l}\right) + d_k^{(s,j)} \sin\left(\frac{k\pi x}{l}\right) \quad (s, j)(IC).$$

Now it is obvious to observe that we have proved the validity of the following assertion.

Theorem 3.2. *If for coefficients $(\frac{c_0^{(i,j)}}{2}, c_1^{(i,j)}, d_1^{(i,j)}, c_2^{(i,j)}, d_2^{(i,j)}, \dots)$ ($1 \leq i \leq I, 1 \leq j \leq J$) functions $\Psi_{(i,j)}(t, x)$ satisfy conditions of Corollary 2.6, then a function $\Psi(t, x) : [0, T[\times [-l, l[\rightarrow R$ defined by*

$$\sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \Psi_{(i,j)}(x, t) \times \chi_{[t_i, t_{i+1}[\times [x_j, x_{j+1}[}(t, x) \quad (3.17)$$

is a weak solution of (3.1)-(3.2).

Example 3.3. *Let consider a linear partial differential equation of the 22 order in two variables*

$$\frac{\partial}{\partial t} \Psi(t, x) = A(t, x) \times \frac{\partial^2}{\partial x^2} \Psi(t, x) + B(t, x) \times \frac{\partial^{22}}{\partial x^{22}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi]) \quad (3.18)$$

with initial condition

$$\Psi(0, x) = \frac{0.015}{2} + 5 \sin(x), \quad (3.19)$$

where

$$A(t, x) = \chi_{[0, \pi[\times [0, \pi[}(t, x) + 1.55 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x)$$

and

$$B(t, x) = 2 \times \chi_{[0, \pi[\times [0, \pi[}(t, x) - 2 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x).$$

The program in MathLab for a solution of (3.18) – (3.19), has the following form:

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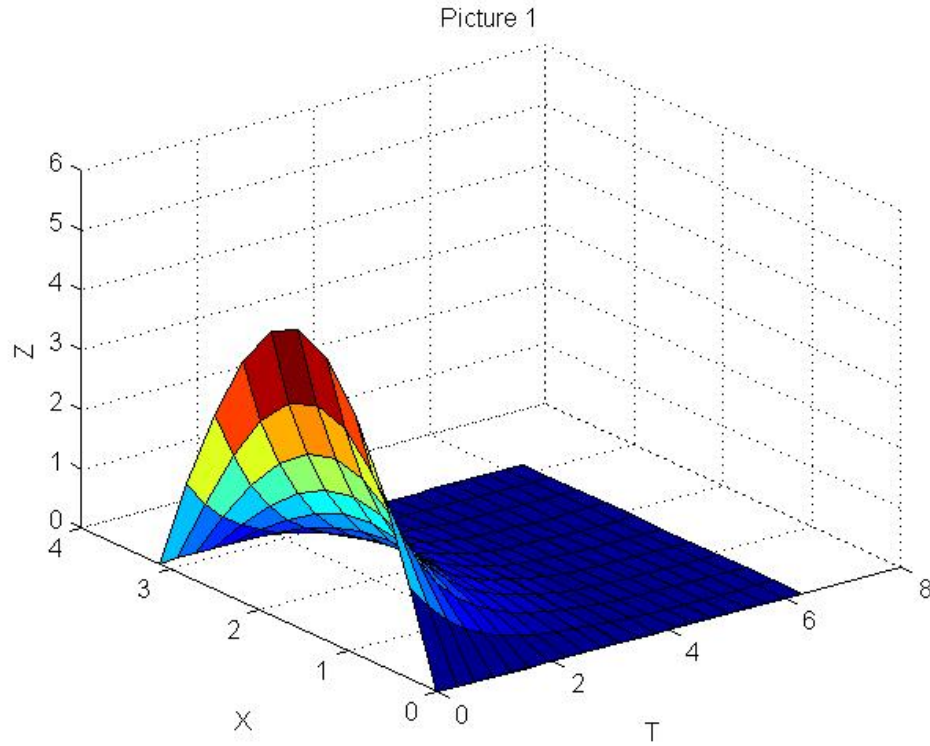
A1 = [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2];
A2 = [0, 1.55, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2];
C1 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
D1 = [5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
A10 = 0; A20 = 0; C10 = 0.015;
for k = 1 : 20
    S1(k) = A10; S2(k) = A20;
    for n = 1 : 10
        S1(k) = S1(k) + (-1)^(n) * A1(2 * n) * k^(2 * n);
        S2(k) = S2(k) + (-1)^(n) * A2(2 * n) * k^(2 * n);
    end
end
for k = 1 : 20
    O1(k) = 0;
    O2(k) = 0;
end
for k = 1 : 20
    for n = 1 : 10
        O1(k) = O1(k) + (-1)^n * A1(2 * n + 1) * k^(2 * n + 1);
        O2(k) = O2(k) + (-1)^n * A2(2 * n + 1) * k^(2 * n + 1);
    end
end
[T1, X1] = meshgrid(0 : (pi/10) : pi, 0 : (pi/10) : pi);
Z1 = 0.5 * C10 * exp(T1 * A10);
for k = 1 : 20
    Z1 = Z1 + C1(k) * exp(T1 * S1(k)) * cos(X1 * k) * cos(T1 * O1(k)) + D1(1) * exp(T1 * S1(k)) * cos(X1 * k) * sin(T1 * O1(k)) +
        D1(k) * exp(T1 * S1(k)) * sin(X1 * k) * cos(T1 * O1(k)) - C1(k) * exp(T1 * S1(k)) * sin(X1 * k) * sin(T1 * O1(k));
end
C20 = exp(pi * (A10 - A20)) * C10;
for k = 1 : 20
    A(k) = exp((S1(k) - S2(k)) * pi) * (C1(k) * cos(O1(k) * pi) + D1(k) * sin(O1(k) * pi));
    B(k) = exp((S1(k) - S2(k)) * pi) * (D1(k) * cos(O1(k) * pi) - C1(k) * sin(O1(k) * pi));
end
for k = 1 : 20
    C2(k) = A(k) * cos(O2(k) * pi) - B(k) * sin(O2(k) * pi);

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D2(k) = B(k) * cos(O2(k) * pi) + A(k) * sin(O2(k) * pi);
end
[T2,X2] = meshgrid(pi : (pi/10) : (2 * pi), 0 : (pi/10) : pi);
Z2 = 0.5 * C20 * exp((T2) * A20);
for k = 1 : 20
    Z2 = Z2 + C2(k) * exp(T2 * S2(k)) .* cos(X2 .* k) .* cos(T2 * O2(k)) + D2(1) * exp(T2 *
S1(k)) .* cos(X2 .* k) .* sin(T2 * O2(k)) +
    D2(k) * exp(T2 * S2(k)) .* sin(X2 .* k) .* cos(T2 * O2(k)) - C2(k) * exp(T2 * S2(k)) .*
sin(X2 .* k) .* sin(T2 * O2(k));
end
surf(T1,X1,Z1)
hold on
surf(T2,X2,Z2)
hold off

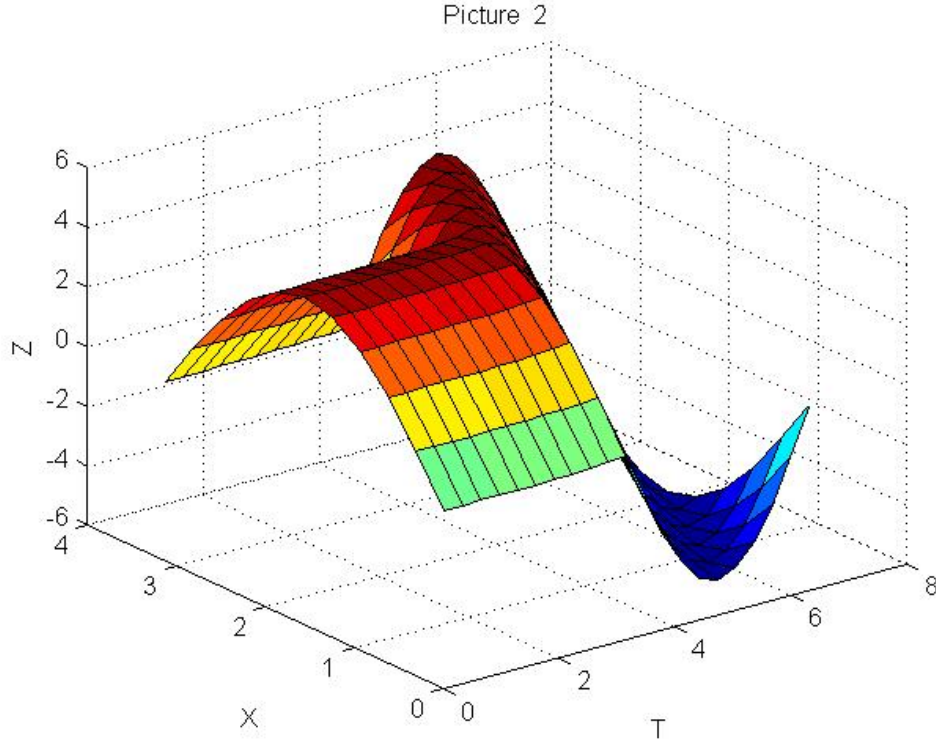
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1. Graphic of the solution of the LPDE-(3.18) with IC-(3.19)).

Example 3.4. Let consider a linear partial differential equation of the 22 order in two variables

$$\frac{\partial}{\partial t} \Psi(t, x) = A(t, x) \Psi(t, x) + B(t, x) \times \frac{\partial^2}{\partial x^2} \Psi(t, x) + C(t, x) \times \frac{\partial^3}{\partial x^3} \Psi(t, x) +$$



2. Graphic of the solution of the LPDE-(3.20) with IC-(3.21)).

$$D(t, x) \times \frac{\partial^{22}}{\partial x^{22}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi[) \quad (3.20)$$

with initial condition

$$\Psi(0, x) = \frac{0.015}{2} + 5 \sin(x), \quad (3.21)$$

where

$$A(t, x) = 1 \chi_{[0, \pi[\times [0, \pi[}(t, x) + 0 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x),$$

$$B(t, x) = \chi_{[0, \pi[\times [0, \pi[}(t, x) + 0 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x),$$

$$C(t, x) = 0 \times \chi_{[0, \pi[\times [0, \pi[}(t, x) + 1 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x)$$

and

$$D(t, x) = 2 \times \chi_{[0, \pi[\times [0, \pi[}(t, x) + 2 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x).$$

The graphical solution of (3.20)-(3.21) can be obtained by MathLab programm used in Example 3.3 for the following data:

$$A1 = [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2];$$

$$A2 = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2];$$

$$C1 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];$$

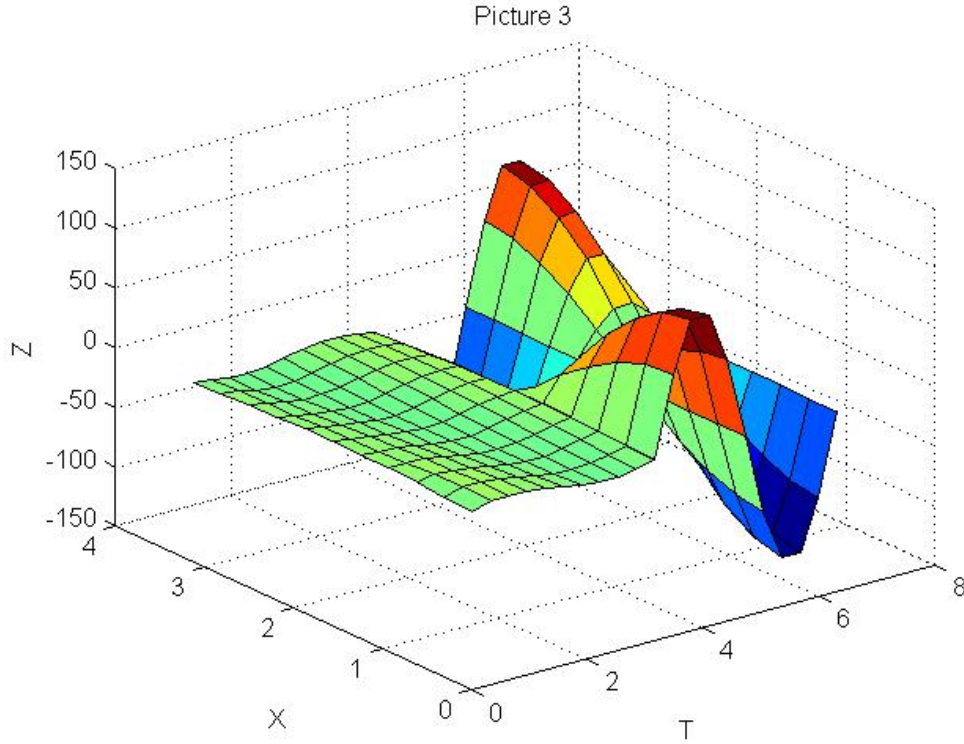
$$D1 = [5, 0];$$

$$A10 = 1; A20 = 0; C10 = 0.015;$$

Example 3.5. Let consider a linear partial differential equation of the 21 order in two variables with constant coefficients

$$\frac{\partial}{\partial t} \Psi(t, x) = A(t, x) \Psi(t, x) + \frac{\partial^2}{\partial x^2} \Psi(t, x) + 2 \frac{\partial^{21}}{\partial x^{21}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi[) \quad (3.22)$$

with initial condition



3. Graphic of the solution of the LPDE-(3.22) with IC-(3.23)).

$$\Psi(0, x) = \frac{0.15}{2} + 5 \sin(x), \quad (3.23)$$

where

$$A(t, x) = 1 \chi_{[0, \pi[\times [0, \pi[}(t, x) + 0 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x).$$

Since

$$1 = 1 \times \chi_{[0, \pi[\times [0, \pi[}(t, x) + 1 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x)$$

and

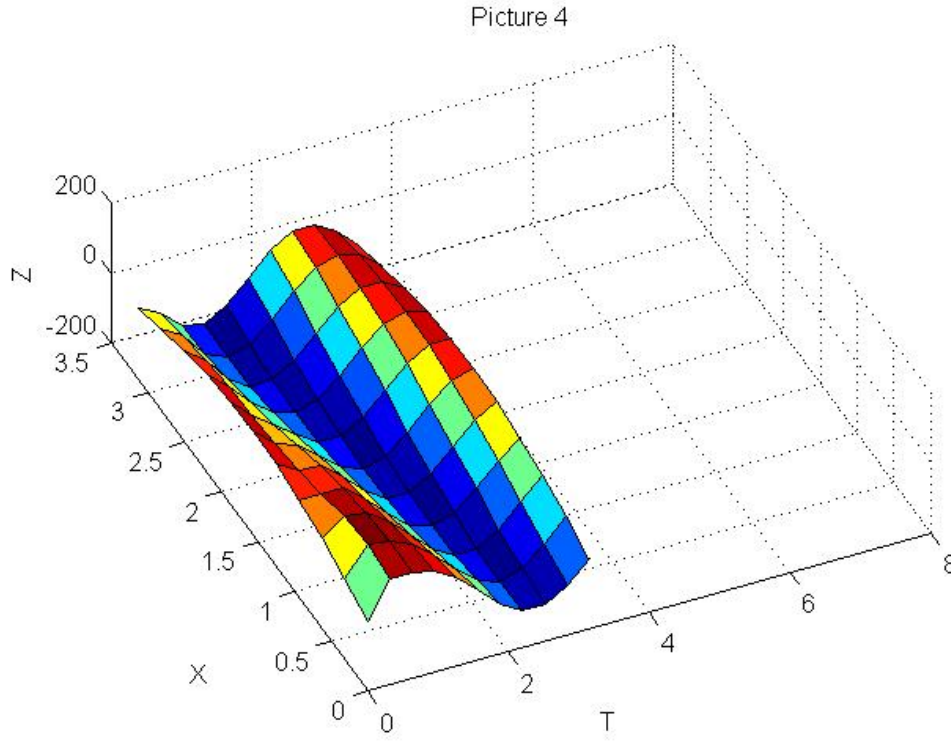
$$2 = 2 \times \chi_{[0, \pi[\times [0, \pi[}(t, x) + 2 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x),$$

the graphical solution of (3.22)-(3.23) can be obtained by MathLab programm used in Example 3.3 for the following data:

$$\begin{aligned}
A1 &= [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\
A2 &= [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\
C1 &= [0, 0]; \\
D1 &= [5, 0]; \\
A10 &= 1; A20 = 0; C10 = 0.15;
\end{aligned}$$

Example 3.6. Let consider a linear partial differential equation of the 21 order in two variables

$$\frac{\partial}{\partial t} \Psi(t, x) = A(t, x) \Psi(t, x) + B(t, x) \times \frac{\partial^2}{\partial x^2} \Psi(t, x) + 100 \frac{\partial^3}{\partial x^3} \Psi(t, x) + 2 \frac{\partial^{21}}{\partial x^{21}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi[) \quad (3.24)$$



4. Graphic of the solution of the LPDE-(3.24) with IC-(3.25)).

with initial condition

$$\Psi(0, x) = \frac{0.015}{2} + 100 \sin(x), \quad (3.25)$$

where

$$A(t, x) = 1 \chi_{[0, \pi[\times [0, \pi[}(t, x) + 0 \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x)$$

and

$$B(t, x) = \chi_{[0, \pi[\times [0, \pi[}(t, x) - \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x).$$

The graphical solution of (3.24)-(3.25) can be obtained by MathLab programm used in Example 3.3 for the following data:

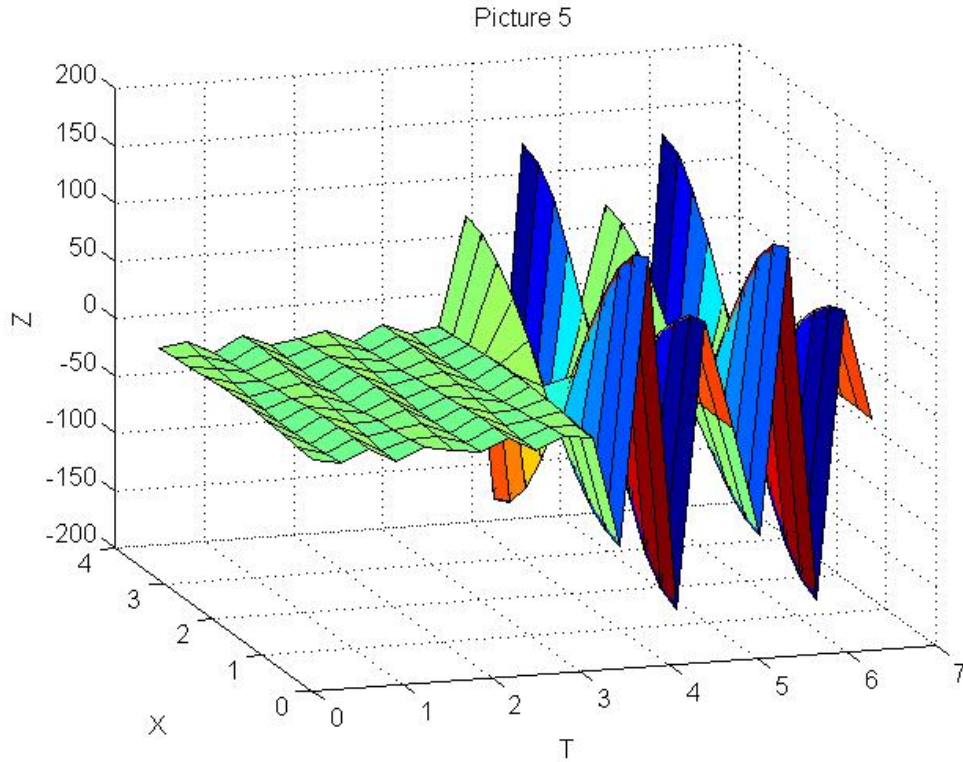
$$\begin{aligned} A1 &= [0, 1, 100, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\ A2 &= [0, -1, 100, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\ C1 &= [0, 0]; \\ D1 &= [100, 0]; \\ A10 &= 1; A20 = 0; C10 = 0.15; \end{aligned}$$

We see that we have no graphic on the region $[\pi, 2\pi[\times [0, \pi[$ which hints us that coefficients of the LPDE (3.24)-(3.25) on that region do not satisfy conditions of Theorem 3.2.

Example 3.7. Let consider a linear partial differential equation of the 21 order in two variables

$$\frac{\partial}{\partial t} \Psi(t, x) = A(x, t) \Psi(t, x) + 50 \frac{\partial^5}{\partial x^5} \Psi(t, x) + \frac{\partial^6}{\partial x^6} \Psi(t, x) + 2 \frac{\partial^{21}}{\partial x^{21}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi[) \quad (3.26)$$

with initial condition



5. Graphic of the solution of the LPDE-(3.26) with IC-(3.27)).

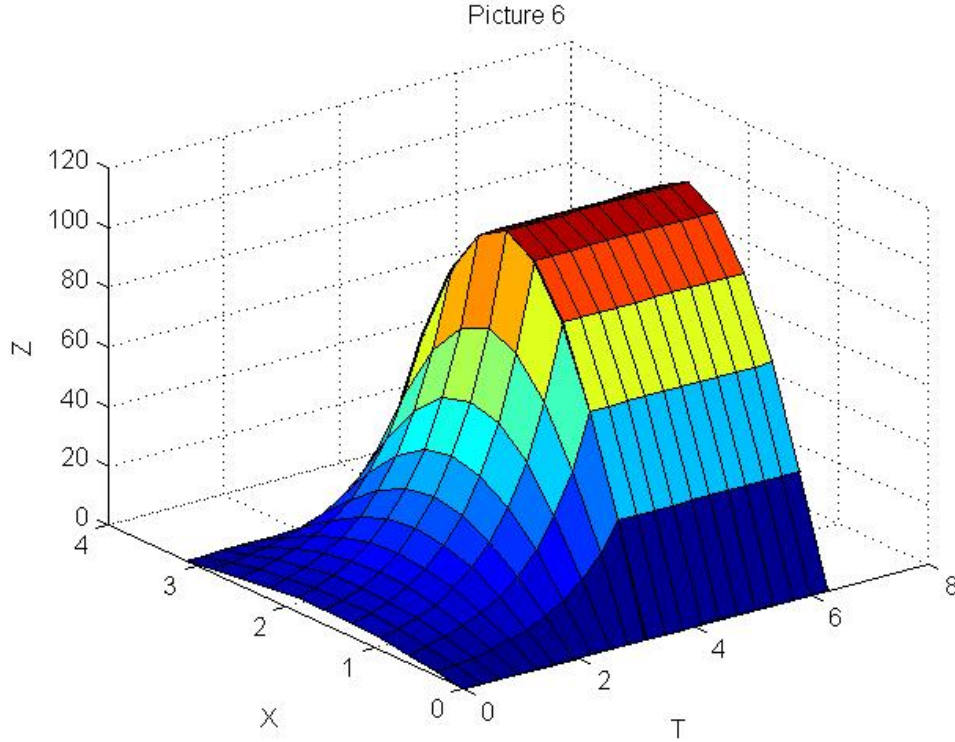
$$\Psi(0, x) = \frac{0.15}{2} + 5 \sin(x), \quad (3.27).$$

where

$$A(t, x) = \chi_{[0, \pi[\times [0, \pi[}(t, x) + 0 \times \chi_{[\pi, 2\pi[\times [0, \pi[}(t, x).$$

The graphical solution of (3.26)-(3.27) can be obtained by MathLab programm used in Example 3.3 for the following data:

$$\begin{aligned} A1 &= [0, 0, 0, 0, 50, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\ A2 &= [0, 0, 0, 0, 50, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0]; \\ C1 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\ D1 &= [7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\ A10 &= 1; A20 = 0; C10 = 0.15; \end{aligned}$$



6. Graphic of the solution of the LPDE-(3.28) with IC-(3.29)).

Example 3.8. Let consider a linear partial differential equation of the 22 order in two variables

$$\frac{\partial}{\partial t} \Psi(t, x) = A(x, t) \Psi(t, x) - \frac{\partial}{\partial x} \Psi(t, x) + 2 \frac{\partial^{22}}{\partial x^{22}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi] \times [0, \pi]) \quad (3.28)$$

with initial condition

$$\Psi(0, x) = \frac{0.015}{2} + 5 \sin(x), \quad (3.29).$$

where

$$A(t, x) = \chi_{[0, \pi] \times [0, \pi]}(t, x) + 0 \times \chi_{[\pi, 2\pi] \times [0, \pi]}(t, x).$$

The graphical solution of (3.28)-(3.29) can be obtained by MathLab programm used in Example 3.3 for the following data:

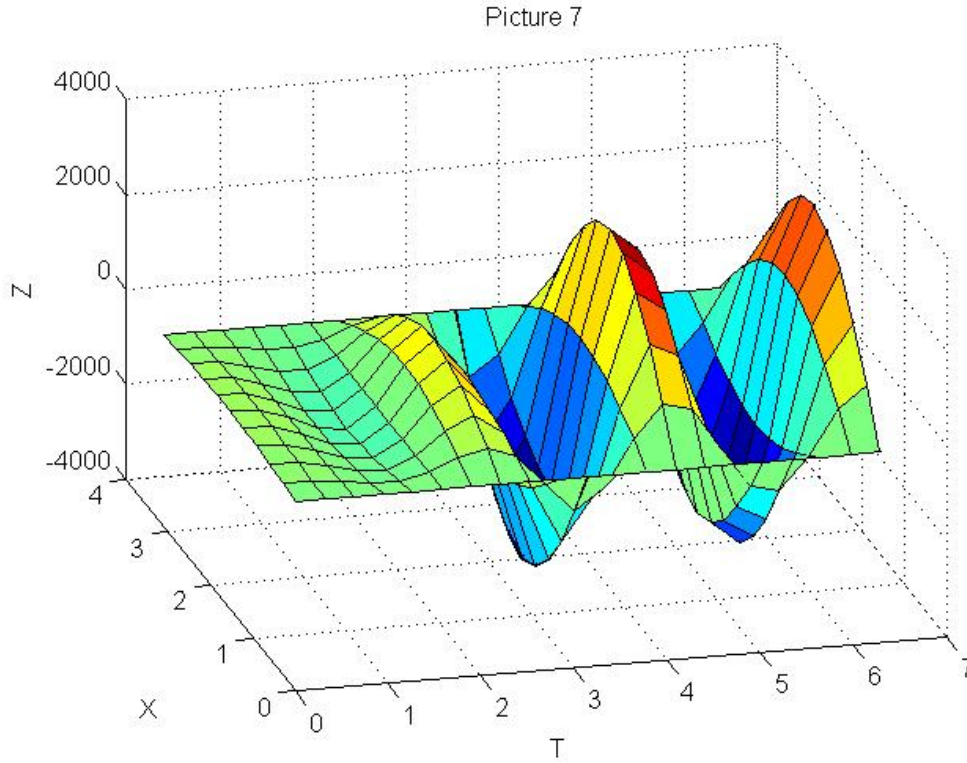
$A1 = [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2];$
 $A2 = [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2];$
 $C1 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];$
 $D1 = [5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];$
 $A10 = 1; A20 = 0; C10 = 0.015;$

Remark 3.9. Notice that for each natural number $M > 1$, one can easily modify the MathLab program described in Example 3.3 for obtaining the graphical solution of the linear partial differential equation (3.1)-(3.2) whose coefficients $(A_n(t, x))_{0 \leq n \leq 2M}$ are real-valued simple step functions on $[0, T[\times [-l, l[$ and f is a trigonometric polynomial on $[-l, l[$.

Remark 3.10. Since each constant c same times is a step function we can use MathLab program described in Example 3.3 for a solution of the linear partial differential equation (2.17)-(2.18) with constant coefficients.

For example, if we consider LPDE

$$\frac{\partial}{\partial t} \Psi(t, x) = -\frac{\partial}{\partial x} \Psi(t, x) + \frac{\partial^3}{\partial x^3} \Psi(t, x) - \frac{\partial^5}{\partial x^5} \Psi(t, x) + \frac{\partial^7}{\partial x^7} \Psi(t, x) + 2 \frac{\partial^{22}}{\partial x^{22}} \Psi(t, x) \quad ((t, x) \in [0, 2\pi[\times [0, \pi]) \quad (3.30)$$



7. Graphic of the solution of the LPDE-(3.30) with IC-(3.1)).

with initial condition

$$\Psi(0, x) = \frac{0.015}{2} + 150 \sin(x), \quad (3.31)$$

then for a solution (3.30)-(3.31), in MathLab programm described in Example 3.3 we must enter the following data:

$$\begin{aligned} A1 &= [-1, 0, 1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2]; \\ A2 &= [-1, 0, 1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2]; \\ C1 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\ D1 &= [150, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\ A10 &= 1; A20 = 0; C10 = 0.015; \end{aligned}$$

Remark 3.11. The approach used for a solution of (3.1)-(3.2) can be used in such a case when coefficients $(A_n(t, x))_{0 \leq n \leq 2M}$ are rather smooth continuous functions on $[0, T] \times [-l, l]$. If we will approximate $(A_n(t, x))_{0 \leq n \leq 2M}$ by real-valued simple step functions, then it is natural to wait that under some "nice restrictions" on $(A_n(t, x))_{0 \leq n \leq 2M}$ the solutions obtained by Theorem 3.2, will give us a "good approximation" of the solution of the required linear partial differential equation of the higher order in two variables with corresponding initial conditions.

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